

Radiation reaction and the self-force for a point mass in general relativity

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A point particle of mass μ moving on a geodesic creates a perturbation h^μ , of the spacetime metric g^0 , that diverges at the particle. Simple expressions are given for the singular μ/r part of h^μ and its quadrupole distortion caused by the spacetime. Subtracting these from h^μ leaves a remainder h^R that is C^1 . The self-force on the particle from its own gravitational field corrects the worldline at $O(\mu)$ to be a geodesic of $g^0 + h^R$. For the case that the particle is a small non-rotating black hole, an approximate solution to the Einstein equations is given with error of $O(\mu^2)$ as $\mu \rightarrow 0$.

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Perturbation analysis provides one approach to understanding the self-force and radiation reaction in general relativity. This begins with a background spacetime metric g^0 which is a vacuum solution of the Einstein equations $G_{ab}(g^0) = 0$. An object of small mass μ then disturbs the geometry by an amount $h^\mu = O(\mu)$ which is determined by the perturbed Einstein equations with the stress-energy tensor of the object being the source,

$$E_{ab}(h^\mu) = -8\pi T_{ab} + O(\mu^2), \quad \mu \rightarrow 0. \quad (1)$$

The superscript μ is a reminder that h^μ is linear in μ . The linear differential operator E_{ab} is defined by

$$E_{ab}(h) = -\frac{\delta G_{ab}}{\delta g_{cd}} h_{cd}, \quad (2)$$

which evaluates to [1]

$$2E_{ab}(h) = \nabla^2 h_{ab} + \nabla_a \nabla_b h - 2\nabla_{(a} \nabla^c h_{b)c} + 2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c \nabla^d h_{cd} - \nabla^2 h). \quad (3)$$

With a solution of Eq. (1) it follows that

$$G_{ab}(g^0 + h^\mu) = 8\pi T_{ab} + O(\mu^2). \quad (4)$$

An integrability condition for Eq. (1) results from the Bianchi identity for $g^0 + h^\mu$ and requires that T be conserved in the background geometry up to $O(\mu^2)$.

Formally, perturbation analysis at the second order is no more difficult than at the first. But the integrability condition for the second order equations is that T be conserved not in the background geometry, but in the first order perturbed geometry. Thus, before solving the second order equations, it is necessary to change the stress-energy tensor in a way which is dependent upon the first order metric perturbations. This correction to T is said to result from the “self-force” on the particle from its own gravitational field and includes the dissipative effects of what is often referred to as “radiation reaction” as well as other nonlinear aspects of general relativity.

To focus on those details of the self-force which are independent of the object’s structure we restrict the object to be a point particle with no spin angular momentum

or other internal structure. The integrability condition at the first order then implies that the worldline Γ of the particle is nearly a geodesic of g^0 , with an acceleration of only $O(\mu)$. But, the integrability condition at the second order presents a difficulty. The particle is to move along a geodesic of $g^0 + h^\mu$, but h^μ scales as μ/r near the particle and is not differentiable on Γ .

Mino, et al. [2] and Quinn and Wald [3] resolve this difficulty with a Green’s function approach to Eq. (1). The formal Hadamard expansion of the Green’s function near the worldline of the particle identifies the “instantaneous” and “tail” parts of h^μ . And, Mino et al. use a matched asymptotic expansion to show that the particle moves along a geodesic of $g^0 + h^\mu_{\text{tail}}$; by construction h^μ_{tail} is differentiable on the worldline as required by the geodesic equation. However, their analysis provides no simple method for determining this tail part.

Alternatively, we resolve the difficulty by finding the source part of the metric perturbation h^S , which consists of the singular μ/r part plus its quadrupole distortion caused by the background geometry. Eqs. (10)-(12) give a simple expression for h^S . Then we show that the remainder, $h^R \equiv h^\mu - h^S$, is C^1 and, using matched asymptotic expansions, that the $O(\mu)$ effect of the self-force adjusts the worldline of the particle to be a geodesic Γ' of $g^0 + h^R$. The consistency of our matched asymptotic expansions with those of Ref. [2] imply that h^R must be equivalent to the “tail” part of the metric perturbation from the Green’s function, up to a gauge transformation and terms of $O(\mu r^2)$, which do not effect the $O(\mu)$ correction to the worldline.

The source field h^S is best described with coordinates in which the background geometry looks as flat as possible near the geodesic Γ . A *normal* coordinate system, $x^a = (t, x, y, z)$, can be found [1] where, on Γ , the metric and its first derivatives match the Minkowski metric, and the coordinate t measures the proper time. Normal coordinates for a geodesic are not unique, and we use particular coordinates introduced by Thorne and Hartle [4] in their discussion of external multipole moments of a vacuum solution of the Einstein equations where

$$g_{ab}^0 = \eta_{ab} + {}_2H_{ab} + O(r^3/\mathcal{R}^3), \quad r/\mathcal{R} \rightarrow 0, \quad (5)$$

with

$${}_2H_{ab}dx^a dx^b = -\mathcal{E}_{ij}x^i x^j (dt^2 + \delta_{kl}dx^k dx^l) + \frac{4}{3}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i dt dx^k. \quad (6)$$

And, \mathcal{E} and \mathcal{B} are spatial, symmetric, tracefree and related to the Riemann tensor evaluated on Γ by $\mathcal{E}_{ij} = R_{titj}$ and $\mathcal{B}_{ij} = \epsilon_i{}^{pq}R_{pqjt}/2$; and, \mathcal{R} is a representative length scale of the background geometry—the smallest of the radius of curvature, the scale of inhomogeneities, and the time scale for changes in curvature along Γ , then \mathcal{E}_{ij} and $\mathcal{B}^q{}_i$ are $O(1/\mathcal{R}^2)$; also (r, θ, ϕ) are defined in the usual way in terms of (x, y, z) ; the indices i, j, k, \dots are spatial and raised and lowered with δ_{ij} .

If a small non-rotating black hole moves along Γ , then its geometry is perturbed by tidal forces,

$$g_{ab}^{\text{pert}} = g_{ab}^{\text{Schw}} + {}_2h_{ab} \quad (7)$$

through terms of $O(r^2/\mathcal{R}^2)$, where ${}_2h$ is a solution of

$$E_{ab}^{\text{Schw}}({}_2h) = 0 \quad (8)$$

with the boundary conditions that the perturbation be well behaved on the event horizon and that ${}_2h \rightarrow {}_2H$ in the *buffer region* [4], where $\mu \ll r \ll \mathcal{R}$. Both ${}_2h$ and ${}_2H$ consist of $\ell = 2$ tensor harmonics in the Regge-Wheeler gauge [5]; the angular dependence is through $x^i x^j \mathcal{E}_{ij}$ and $\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i$. For $r \ll \mathcal{R}$, ${}_2h$ is governed by a wave equation with a potential barrier. In the time independent limit this admits an analytic solution [5]

$${}_2h_{ab}dx^a dx^b = -\mathcal{E}_{ij}x^i x^j [(1 - 2\mu/r)^2 dt^2 + dr^2 + (r^2 - 2\mu^2)(d\theta^2 + \sin^2 \theta d\phi^2)] + \frac{4}{3}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i (1 - 2\mu/r) dt dx^k \quad (9)$$

which is well behaved on the event horizon and matches ${}_2H$ when $\mu \ll r$. See Ref. [6] for the case of a small black hole in the vicinity of a larger black hole.

Time dependence of ${}_2H$ induces a quadrupole moment on the black hole, but the resulting acceleration of the world line is smaller than $O(\mu/\mathcal{R}^2)$. Generally, the timescale of ${}_2H$ is \mathcal{R} and corresponds to a low frequency for the black hole, $\omega\mu = O(\mu/\mathcal{R}) \ll 1$. And two independent solutions for the metric perturbation ${}_2h$ are standing waves very near $r = 2\mu$ but behave as r^2 and $1/r^3$, for $\mu \ll r \ll \mathcal{R}$. The proper solution is a traveling wave into the hole created by a linear combination of these two independent solutions having comparable magnitudes when $r \approx 2\mu$. Thus, for $r \gg \mu$ this linear combination is approximately given by Eq. (9), which scales as r^2 , plus a $1/r^3$ contribution from the induced quadrupole moment, $\mathcal{I}^{ab} = O(\mu^5/\mathcal{R}^2)$, stemming from the time dependence. This contribution to the quadrupole field couples to the background octupole field and accelerates the worldline [7] by $\sim \mathcal{E}^i{}_{ab}\mathcal{I}^{ab}/\mu = O(\mu^4/\mathcal{R}^5)$ which is too small to be important in this analysis.

In the buffer region, where $\mu \ll r \ll \mathcal{R}$, the geometry of a point particle moving through the background should be equally well described either by the background metric perturbed by μ , or by the leading μ/r terms of the Schwarzschild metric perturbed by weak tidal forces. In this region, then, the background metric perturbation h^μ is approximately the part of g^{pert} which is linear in μ ; and this part is the source field,

$$h^S \equiv {}_0h^\mu + {}_2h^\mu, \quad (10)$$

where

$${}_0h_{ab}^\mu dx^a dx^b = 2\mu/r(dt^2 + dr^2) \quad (11)$$

is the μ/r part of the Schwarzschild metric, and

$${}_2h_{ab}^\mu dx^a dx^b = \frac{4\mu}{r}\mathcal{E}_{ij}x^i x^j dt^2 - \frac{8\mu}{3r}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i dt dx^k \quad (12)$$

is the $\mu r/\mathcal{R}^2$ part of ${}_2h$ from Eq. (9). A split of h^μ , from Eq. (1), into two pieces,

$$h^\mu = h^S + h^R, \quad (13)$$

reveals just how accurately h^S approximates h^μ by consideration of the remainder h^R . From Eq. (1)

$$E_{ab}(h^R) = -E_{ab}(h^S) - 8\pi T_{ab}. \quad (14)$$

And, direct evaluation shows that

$$E_{ab}(h^S) + 8\pi T_{ab} = O(\mu/\mathcal{R}^3), \quad r \rightarrow 0, \quad (15)$$

for a point particle stress tensor, and the source of Eq. (14) is finite but not continuous at $r = 0$. This last result is understandable—in an expansion of $E_{ab}(h^S)$ in powers of μ and $1/\mathcal{R}$, all of the μ/\mathcal{R}^2 terms would also appear in a similar expansion of $E_{ab}^{\text{Schw}}({}_2h)$. And this latter expansion is zero from the definition of ${}_2h$. Thus, the source of Eq. (14) is $O(\mu/\mathcal{R}^3)$ as $r \rightarrow 0$.

The solution of Eq. (14) for h^R with reasonable boundary conditions is C^1 . If the metric is sufficiently differentiable and there are no unreasonable boundary conditions at large distances, then h^R is differentiable away from the geodesic Γ . For if it were not, then the discontinuities in the derivatives would propagate along the characteristics of the hyperbolic operator E_{ab} and would have originated either on Γ or on some boundary; we consider such discontinuities emanating from a boundary to be unreasonable boundary conditions. And, in a neighborhood of Γ the geometry can be smoothly mapped to flat spacetime with the operator E_{ab} (in the Lorentz gauge) being smoothly mapped to the flat spacetime wave operator which, when integrated twice, smoothes a slowly changing and finite but discontinuous (on Γ) source into a C^1 solution. Thus the difference between h^μ and h^S is a C^1 tensor field h^R .

The split of h^μ into h^S and h^R contains some arbitrariness. Any piece of $O(\mu r^2/\mathcal{R}^3)$ or with higher powers of r/\mathcal{R} can be moved between h^S and h^R without affecting either Eq. (14), the differentiability of h^R , or the $O(\mu)$ effect of the self-force which changes the worldline to be a geodesic Γ' of $g^0 + h^R$. Furthermore, a gauge transformation $y^a = x^a + \xi^a$ for any ξ^a that is $O(r^3/\mathcal{R}^2)$ gives a new normal coordinate system; and we state without details that the corresponding change in h^S is only $O(\mu r^2/\mathcal{R}^3)$ with vanishing derivatives on the worldline. Thus Γ' is independent of the normal coordinate system in use.

But, it is not sufficient to have h^S just consist of ${}_0h^\mu$. If ${}_2h^\mu$ were not included, then $E_{ab}(h^S) + 8\pi T_{ab}$ would be singular $\sim \mu/r\mathcal{R}^2$ as $r \rightarrow 0$. The resulting h^R would not be differentiable on the worldline, and some version of averaging around the particle would be required to make sense of the effects of the self-force. Thus, it is necessary to include ${}_2h^\mu$ as part of h^S .

Above, we mentioned that the worldline of a small particle through the background is a geodesic of $g^0 + h^R$ when the $O(\mu)$ corrections are included. We now justify this statement by replacing the particle with a small, non-rotating black hole and considering a sequence of metrics $g(\mu)$ which are solutions of the vacuum Einstein equations, and parameterized by μ with $g(0) = g^0$. Our focus is on the $O(\mu)$ behaviors of $g(\mu)$ and the worldline in the limits $\mu \rightarrow 0$ and the worldline approaching Γ . Matching asymptotic expansions in the buffer region provides both the correction to the worldline as well as an approximation to $g(\mu)$, Eq. (19) below, which is uniformly valid with an error of $O(\mu^2/\mathcal{R}^2)$, as $\mu \rightarrow 0$.

This matching relies upon a choice of coordinates similar to that of Eq. (5) but for the metric $g^0 + h^R$ being expanded about its geodesic Γ' ; these “primed” normal coordinates differ from the original by $O(\mu)$. In this discussion of matching, r is now associated with Γ' and the primed coordinates.

In the buffer region $g(\mu)$ is best described in a fashion introduced by Thorne and Hartle [4] as a sum of terms of positive powers of the small numbers μ/r and r/\mathcal{R} ,

$$\begin{aligned} g(\mu) \sim & \eta' \quad \& \quad 0 \quad \& \quad {}_2H' \quad \& \quad \cdots \\ & \& \quad \mu/r \quad \& \quad \mu/\mathcal{R} \quad \& \quad \mu r/\mathcal{R}^2 \quad \& \quad \cdots \\ & \& \quad \mu^2/r^2 \quad \& \quad \mu^2/r\mathcal{R} \quad \& \quad \mu^2/\mathcal{R}^2 \quad \& \quad \cdots \\ & \& \quad \cdots, \end{aligned} \quad (16)$$

where $\&$ means “and a term of the form ...” In this tableau, part of any term can be moved diagonally, up and to the left, to be absorbed into a dominating term with the same r behavior.

Just outside the buffer region, where $\mu \ll r < \mathcal{R}$, $g(\mu)$ is approximately the background geometry perturbed by μ . The top row of the tableau consists of the expansion of g^0 about Γ' in powers of r/\mathcal{R} along with some parts of a similar expansion of h^R . The μ/\mathcal{R} and $\mu r/\mathcal{R}^2$ terms from h^R are absorbed into the first two elements of the top row, which are then displayed as η' and zero because the primed coordinates are normal. The r^2/\mathcal{R}^2 term of

the tableau is set to ${}_2H'$, a tensor whose components in the primed coordinates are the same as those of ${}_2H$ in the original coordinates. The difference between ${}_2H'$ and ${}_2H$ is of $O(\mu r^2/\mathcal{R}^3)$ and is subtracted from the fourth term in the second row of the tableau. The rest of the first row consists precisely of the corresponding terms in the expansion of g^0 . And the remainder of the expansion of h^R , from the $\mu r^2/\mathcal{R}^3$ term outward, contribute to the corresponding terms of the second row. Thus the entire first row along with the fourth and greater terms of the second row sums to $g^0 + h^R$.

Just inside the buffer region, where $2\mu < r \ll \mathcal{R}$, $g(\mu)$ is approximately the Schwarzschild geometry perturbed by background tidal forces. The first column of the tableau, containing no \mathcal{R} , is an expansion of the Schwarzschild geometry in powers of μ/r . The second column, linear in $1/\mathcal{R}$, sums to a dipole perturbation of the Schwarzschild geometry. But, the top element of the second column is zero, so all elements of the second column are zero. The top term in the third column, ${}_2H'$, when added to the rest of the third column gives ${}_2h'$, the quadrupole perturbation of the black hole caused by tidal forces. Thus, the first three elements of the top row determine the entire first three columns of this tableau by the expansions of g^{Schw} and ${}_2h'$ in powers of μ/r .

Now, $g^0 + h^R$ is an accurate approximation of $g(\mu)$ when $\mu \ll r$, and $g^{\text{Schw}} + {}_2h'$ is an accurate approximation when $r \ll \mathcal{R}$ and with g^{Schw} centered on Γ' . These approximations overlap in the buffer region where

$$g_{ab}^0 + h_{ab}^R = \eta'_{ab} + {}_2H'_{ab} + O(r^3/\mathcal{R}^3), \quad (17)$$

and

$$g_{ab}^{\text{Schw}} + {}_2h'_{ab} = \eta'_{ab} + {}_2H'_{ab} + O(\mu/r) \quad (18)$$

match asymptotically. In the restricted region $\mu/r \ll r^2/\mathcal{R}^2 \ll 1$, the displayed term ${}_2H' = O(r^2/\mathcal{R}^2)$ is small yet much larger than either of the remainder terms, $O(r^3/\mathcal{R}^3)$ and $O(\mu/r)$, as $\mu/\mathcal{R} \rightarrow 0$. This is the hallmark of matched asymptotic expansions. If the worldline of the black hole were not a geodesic of $g^0 + h^R$ then Eq. (17) would necessarily contain a term of $O(r/\mathcal{R})$. But an explicit $O(r/\mathcal{R})$ term in Eq. (18) would be the dominant term of a dipole perturbation, and such a dipole vacuum perturbation of the Schwarzschild geometry is always removable by a gauge transformation [5]. Thus, this asymptotic matching is only successful when the worldline Γ' is a geodesic of $g^0 + h^R$ up to an acceleration of $O(\mu^2/\mathcal{R}^3)$ in the limit that $\mu/\mathcal{R} \rightarrow 0$.

A concise description of this matched geometry is

$$\begin{aligned} g_{ab}(\mu) = & (g_{ab}^0 + h_{ab}^R) + (g_{ab}^{\text{Schw}} + {}_2h'_{ab}) \\ & - (\eta'_{ab} + {}_2H'_{ab}) + O(\mu^2/\mathcal{R}^2), \quad \mu/\mathcal{R} \rightarrow 0. \end{aligned} \quad (19)$$

For $r \ll \mathcal{R}$, the first and third terms on the right nearly cancel and give $g(\mu) \approx g^{\text{Schw}} + {}_2h'$, the first three columns of the tableau. For $\mu \ll r$ the second and third terms on the right yield $h^{S'} + O(\mu^2/r^2)$. And $g(\mu) \approx g^0 + h^R + h^{S'}$,

the top two rows of the tableau. The error in this approximation comes from the $\mu^2 r/\mathcal{R}^3$ term in the tableau, the dominant term not fixed in the above discussion. This term and the rest of the third row sum to a contribution of $O(\mu^2/\mathcal{R}^2)$. Thus, the matched asymptotic expansions give an approximation to $g(\mu)$ in Eq. (19) which is uniformly valid with error of $O(\mu^2/\mathcal{R}^2)$ as $\mu \rightarrow 0$.

An application of this approach, in conjunction with Fourier-harmonic decomposition, determines the $O(\mu)$ corrections to geodesic motion for a small non-rotating mass in orbit about a much larger non-rotating black hole. First, Eq. (1) is solved for h^μ using the usual metric perturbation analysis of the Regge-Wheeler formalism [5]. This involves decomposing T into its Fourier-harmonic modes; then, the inhomogeneous Regge-Wheeler or Zerilli equation [5] is integrated numerically to determine the radial dependence of the modes of h^μ . And, given the appropriate coordinate transformation, the components of h^S can be transformed from Eqs. (11) and (12) to the usual Schwarzschild coordinates and then numerically decomposed into their Fourier-harmonic modes. Now, h^R and its derivatives can be constructed as the sum over modes of the difference between h^μ and h^S . And, the $O(\mu)$ effect on the worldline of the small mass can be calculated as a change from a geodesic of g^0 to a geodesic of $g^0 + h^R$.

This application depends upon the transformation between the normal coordinates of Eq. (5) and Schwarzschild coordinates. Manasse and Misner [8] give a prescription for finding Fermi-normal coordinates for any geodesic; and Zhang [9] gives a gauge transformation from these to the coordinates of Eq. (5). But Eq. (5) only determines the normal coordinates near the worldline and up to terms of $O(r^4/\mathcal{R}^3)$. So, the normal coordinates must be extended to cover the Schwarzschild manifold for all radii in the vicinity of the orbit to make the mode decomposition of h^S possible. Fortunately, the details of this extension are not important because h^R is known to be differentiable and, therefore, easily describable in terms of a sum over its modes. While the amplitude of each individual mode of h^S does depend upon the extension of the coordinates away from the worldline, the reconstruction of h^R near the worldline by the sum over modes is independent of this extension.

This analysis shows that simply removing the divergent monopole part ${}_0h^\mu$ from the metric perturbation h^μ leaves a nondifferentiable remainder. But, if the quadrupole distortion of the monopole is also removed from h^μ , then the remainder, $h^R = h^\mu - {}_0h^\mu - {}_2h^\mu$, is

differentiable and suitable for calculating $O(\mu)$ effects on the worldline.

The octupole term ${}_3h^\mu$ can also be removed from h^μ . Thorne and Hartle [4] extend the coordinates in Eq. (5) to include $O(r^3/\mathcal{R}^3)$ terms explicitly, their Eqs. (A1) and (A2). The time independent solution of $E_{ab}^{\text{Schw}}({}_3h) = 0$ which is well behaved on the event horizon and properly matches the $O(r^3/\mathcal{R}^3)$ terms of Ref. [4] is

$$\begin{aligned} {}_3h_{ab}dx^a dx^b = & -\frac{1}{3}\mathcal{E}_{ijk}x^i x^j x^k \left[\left(1 - \frac{2\mu}{r}\right)^2 \left(1 - \frac{\mu}{r}\right) dt^2 \right. \\ & + \left(1 - \frac{\mu}{r}\right) dr^2 + \left(r^2 - 2\mu r + \frac{4\mu^3}{5r}\right) (d\theta^2 + \sin^2 \theta d\phi^2) \\ & \left. + \frac{2}{3}\epsilon_{kpq}\mathcal{B}^a_{ij}x^p x^i x^j \left(1 - \frac{2\mu}{r}\right) \left(1 - \frac{4\mu}{3r}\right) dt dx^k \right]. \end{aligned} \quad (20)$$

And, ${}_3H$ and ${}_3h^\mu$ are the r^3/\mathcal{R}^3 and $\mu r^2/\mathcal{R}^3$ parts of ${}_3h$. If ${}_3h^\mu$ is also removed from h^μ then the remainder is C^2 , the matching is extended through the $1/\mathcal{R}^3$ terms, but the overall error of the matched geometry is still $O(\mu^2/\mathcal{R}^2)$ and there is no effect on the worldline.

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